Estimating the post-measurement state

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We study generalized measurements (POVM measurements) on a single d-level quantum system which is in a completely unknown pure state, and derive the best estimate of the post-measurement state. The mean post-measurement estimation fidelity of a generalized measurement is obtained and related to the operation fidelity of the device. This illustrates how the information gain about the post-measurement state and the corresponding state disturbance are mutually dependent. The connection between the best estimates of the pre- and post-measurement state is established and interpreted. For pure generalized measurements the two states coincide.

PACs number(s): 03.67.-a, 03.65.Ta

There are two important properties in which measurements on a quantum system differ from measurements in classical physics: Even if a finite number of identical copies of a system are available, it is in general impossible to obtain complete information about the state of the system. Furthermore, information can be extracted from a quantum system only at the cost of disturbing it.

These aspects are studied in the framework of quantum estimation theory which has recently attracted much interest. It plays an important role in quantum data processing in the context of quantum information and computing. A typical topic is the determination of the optimal fidelity of the estimated quantum state from N identically prepared copies of the quantum system [1]. Algorithms for constructing an optimal positive operator valued measure (POVM measurement) were discussed in [2]. Adaptive projection measurements were treated in [3]. A related subject to the present discussion is the tension between information gain and disturbance [4]. The balance between the mean operation fidelity and the estimation fidelity of the pre-measurement state has been studied by Banaszek [5]. We will come back to his results later.

The purpose of this Letter is studying the estimation of the post-measurement state. Suppose a generalized measurement (POVM measurement) is performed on a single d-level system of pure but otherwise completely unknown quantum state. Knowing the measurement result and the specifications of the measurement, what is the best estimate of the post-measurement state and what is the corresponding highest fidelity? Of all measurements granting a certain estimation fidelity, which is the one with the lowest disturbance? And finally, how are the best estimations of the pre- and post-measurement states related? All these questions will be answered below in closed analytical forms.

Situations in which it is important to guess the "post-measurement state" are known from everyday life. Medical inspections with x-rays, radioactive chemicals etc. are invasive measurements as quantum measurements in general are. The more information such inspections provide the more damage they cause. No copy of the patient is available. The patient's state is therefore to be estimated on the basis of a single run inspection whereby the doctor has to decide about the strength of his intervention in choosing a balance between information gain and disturbance. Since, furthermore, any subsequent medical treatment must take into account that an unavoidable disturbance has happened, it has to be adjusted to the post- and not to the pre-inspection state.

There are quantum informatic setups exhibiting such characteristic traits. A typical example is a sequence of generalized measurements aiming at the monitoring of the state evolution of a single quantum system [6]. An important strategy to improve the information and to diminish the disturbance is to adjust the parameters of each forthcoming generalized measurement to the expected pre-measurement state [7]. To this end, the post-measurement state of the previous measurement must be estimated. We will not work out this example, but rather turn to the post-measurement state in general.

A given generalized measurement is described by a set of n operators M_s, where the index $s = 1, \ldots, n$ labels the possible readouts of the measurement. These measurement operators, also called Kraus-operators, act on the quantum state of the measured system. One may think of a d-level system. The readout s will in general not correspond to one of these levels, contrary to typical projective measurements. The pure pre-measurement state $|\psi\rangle$ of the system is changed by a generalized measurement with outcome s into the conditional post-measurement state

$$|\psi^{(s)}\rangle = \frac{M_s|\psi\rangle}{\sqrt{\langle\psi|M_s^\dagger M_s|\psi\rangle}} \; . \qquad (1)$$

Obviously, $|\psi^{(s)}\rangle$ will always depend on the initial state $|\psi\rangle$ unless the rank of M_s is 1. Therefore the post-measurement state remains in general unknown if $|\psi\rangle$ is unknown, and can only be estimated. The probability for the measurement result s to occur is given by

$$p_s = \langle\psi|E_s|\psi\rangle \, , \qquad (2)$$

where the operators E_s are defined by

$$E_s := M_s^\dagger M_s \; . \qquad (3)$$

They are positive operators satisfying a completeness relation $\sum_{s=1}^{n} E_s = \mathbb{1}$ which guarantees $\sum_{s=1}^{n} p_s = 1$ for the probabilities. The set $\{E_s\}$ is called a positive-operator-valued measure (POVM) and the individual operators E_s are also known as POVM elements or effects.

To prepare later calculations we introduce the spectral decomposition of E_s

$$E_s = \sum_{i=1}^{d} a_i^{(s)} |r_i^{(s)}\rangle\langle r_i^{(s)}| \ . \tag{4}$$

$a_i^{(s)}$ are the positive eigenvalues. The eigenvectors $\{|r_i^{(s)}\rangle\}$ form an orthonormal basis. Due to the polar decomposition theorem (cf. e.g. [8]), we may split the measurement operator M_s into a product of a unitary operator U_s and the square root of E_s

$$M_s = U_s \sqrt{E_s} \ . \tag{5}$$

This implies

$$M_s M_s^\dagger = U_s E_s U_s^\dagger \ . \tag{6}$$

Thus the positive operators $M_s M_s^\dagger$ and E_s have the same eigenvalues $a_i^{(s)}$ and the diagonal representation of $M_s M_s^\dagger$ becomes

$$M_s M_s^\dagger = \sum_{i=1}^{d} a_i^{(s)} |l_i^{(s)}\rangle\langle l_i^{(s)}| \ . \tag{7}$$

The eigenvectors $|l_i^{(s)}\rangle = U_s |r_i^{(s)}\rangle$ form again an orthonormal basis. Herewith and with the help of eqs.(4) and (5) we obtain as result the useful *bi-orthogonal expansions* of the unitary operators U_s and of the measurement operators M_s:

$$U_s = \sum_{i=1}^{d} |l_i^{(s)}\rangle\langle r_i^{(s)}| \tag{8}$$

$$M_s = \sum_{i=1}^{d} \sqrt{a_i^{(s)}} |l_i^{(s)}\rangle\langle r_i^{(s)}| \ . \tag{9}$$

$|l_i^{(s)}\rangle$ and $|r_i^{(s)}\rangle$ are the l.h.s. and r.h.s. eigenvectors of M_s, respectively. The number of non-zero eigenvalues $\sqrt{a_i^{(s)}}$ equals the rank of M_s.

Based on this we can now move to the problems of quantum state estimation. We assume a *single* d-level quantum system prepared in a completely unknown pure pre-measurement state $|\psi\rangle$. A particular generalized measurement specified by the known set $\{M_s\}$ of operators is performed with measurement result s which is read off. What is the optimal strategy for the estimation of the post-measurement state $|\psi^{(s)}\rangle$ prepared by the measurement? It is worthwhile to emphasize that the only data available for the estimation are the set $\{M_s\}$ specifying the measurement and the value s of the actual readout.

If the state $|\chi^{(s)}\rangle$ is proposed as an estimate of the unknown post-measurement state $|\psi^{(s)}\rangle$, the fidelity

$$f_s = |\langle\chi^{(s)}|\psi^{(s)}\rangle|^2 = \frac{1}{p_s} |\langle\chi^{(s)}|M_s|\psi\rangle|^2 \tag{10}$$

is a measure of the quality of the estimation. The fidelity $\overline{f}$ averaged over all measurement outcomes reads: $\overline{f} = \sum_{s=1}^{n} f_s p_s$. The mean estimation fidelity $G_{\text{post}}(\chi)$ in case the in-going (pre-measurement) state is *completely unknown*, is the result of an integration over all possible states $|\psi\rangle$:

$$G_{\text{post}}(\chi) := \int \overline{f} d\psi = \int d\psi \sum_{s=1}^{n} \langle\chi^{(s)}|M_s|\psi\rangle\langle\psi|M_s^\dagger|\chi^{(s)}\rangle \ , \tag{11}$$

with respect to the normalized unitary invariant measure on the state space, yielding:

$$G_{\text{post}}(\chi) = \frac{1}{d} \sum_{s=1}^{n} \langle\chi^{(s)}|M_s M_s^\dagger|\chi^{(s)}\rangle \ . \tag{12}$$

By virtue of eq.(7), each component in the sum over s in (12) is maximized if $|\chi^{(s)}\rangle$ is chosen to be the eigenvector $|l_{\max}^{(s)}\rangle$ of $M_s M_s^\dagger$ of the maximum eigenvalue $a_{\max}^{(s)}$. For the measurement result s the *best estimate of the post-measurement state* is therefore given by

$$|\chi_{\text{post}}^{(s)}\rangle = |l_{\max}^{(s)}\rangle. \tag{13}$$

In case of degeneracy of the greatest eigenvalue $a_{\max}^{(s)}$, any state vector from the corresponding eigenspace represents an optimal estimation of the post-measurement state. The maximum value of $G_{\text{post}}(\chi)$ reads

$$G_{\text{post}} = \frac{1}{d} \sum_{s=1}^{n} a_{\max}^{(s)} \ . \tag{14}$$

G_{post} is the *mean post-measurement estimation fidelity.* $|\chi_{\text{post}}^{(s)}\rangle$ and G_{post} are determined solely by the operators M_s which specify the generalized measurement.

We now address the question, how G_{post} is related to the *mean operation fidelity* F which describes how much the state after the measurement resembles the original one. The larger the value F of a measurement is, the weaker is its disturbing influence. Arguing as above, F is obtained from eq.(11) if we replace $|\chi^{(s)}\rangle$ by $|\psi\rangle$:

$$F = \int d\psi \sum_{s=1}^{n} |\langle\psi|M_s|\psi\rangle|^2 \ . \tag{15}$$

It may be rewritten as [5]:

$$F = \frac{1}{d(d+1)} \Big(d + \sum_{s=1}^{n} |\mathrm{tr} M_s|^2 \Big) \ . \tag{16}$$

To derive a relation between $G_{\rm post}$ and F, it is useful to first relate $G_{\rm post}$ to the estimation fidelity of the pre-measurement state. Denoting this estimate by $|\chi^{(s)}\rangle$, the corresponding mean estimation fidelity, in analogy to $G_{\rm post}(\chi)$ of eq.(11), reads:

$$G_{\rm pre}(\chi) = \int d\psi \sum_{s=1}^{n} p_s |\langle \chi^{(s)}|\psi\rangle|^2 \tag{17}$$

which may be rewritten according to Banaszek [5] as

$$G_{\rm pre}(\chi) = \frac{1}{d(d+1)} \left(d + \sum_{s=1}^{n} \langle \chi^{(s)}|E_s|\chi^{(s)}\rangle \right) \ . \tag{18}$$

The optimum pre- and post-measurement fidelities are closely related. For a given measurement result s, the best estimate $|\chi^{(s)}_{\rm pre}\rangle$ of the pre-measurement state is the one which maximizes the corresponding component in the sum in eq.(18). Because of eq.(4), it is given by the eigenvector $|r^{(s)}_{\rm max}\rangle$ of E_s belonging to the maximum eigenvalue [5,9]. But this eigenvalue is again $a^{(s)}_{\rm max}$. The *best estimate of the pre-measurement state* related to the outcome s is therefore

$$|\chi^{(s)}_{\rm pre}\rangle = |r^{(s)}_{\rm max}\rangle \ . \tag{19}$$

We denote the corresponding maximum value of $G_{\rm pre}(\chi)$ by $G_{\rm pre}$ and call it the *mean pre-measurement estimation fidelity.* Comparing it to the form (14) of $G_{\rm post}$, we obtain the simple new relationship:

$$G_{\rm pre} = \frac{1}{d+1}(1 + G_{\rm post}) \ . \tag{20}$$

This result allows us to transcribe Banaszek's constraint [5] between F and $G_{\rm pre}$ into a constraint relating F and $G_{\rm post}$:

$$\sqrt{(d+1)F - 1} \le \sqrt{G_{\rm post}} + \sqrt{(d-1)(1-G_{\rm post})} \ . \tag{21}$$

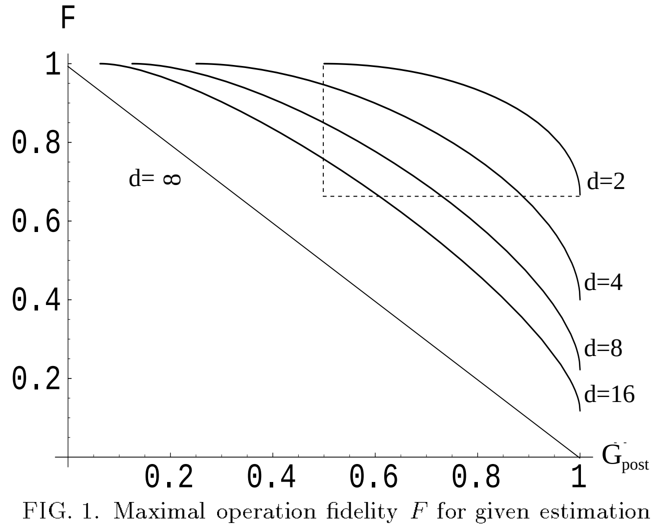

FIG. 1. Maximal operation fidelity F for given estimation fidelity $G_{\rm post}$ of the post-measurement state in dimensions $d = 2, 4, 8, 16, \infty$.The dashed lines mark for dimension $d = 2$ the domain for possible combinations of F and $G_{\rm post}$.

To illustrate how state disturbance and information gain are related for the post-measurement situation, we display the domain of possible combination of F and $G_{\rm post}$ in the $G_{\rm post}$-F plane. If the system is not influenced at all, the measurement has the operation fidelity $F = 1$. In this case the guess of the pre- and post-measurement state is totally random which amounts to $G_{\rm pre} = G_{\rm post} = 1/d$. On the other hand there are measurements which allow to predict the post-measurement state exactly (e.g. projection measurements), i.e., with maximum fidelity $G_{\rm post} = 1$. This leads via eq.(20) to $G_{\rm pre} = 2/(d+1)$. This result for $G_{\rm pre}$ has also been obtained in [1,4–6]. It is known [2] that it corresponds to $F = 2/(d+1)$. To summarize, the domain of possible combinations $(G_{\rm post}, F)$ is limited by $1/d \le G_{\rm post} \le 1$ and $2/(d+1) \le F \le 1$ as well as by the inequality (21). The boundaries of the domain are indicated in Fig. 1 for $d = 2$, including the dashed lines. In this domain, every particular generalized measurement $\{M_s\}$ corresponds to a point. Its position illustrates to what extent the information about the outgoing (post-measurement) state is gained at the cost of disturbing the in-going (pre-measurement) one. Large values of F combined with large values of $G_{\rm post}$ characterize the most optimal type of generalized measurement. For increasing dimension d of the state space all types of measurements become less advantageous (cf. Fig. 1).

To complete this discussion we return to the question: What type of generalized measurements apart from projection measurements make it possible to know the post-measurement state $|\psi^{(s)}\rangle$ exactly? As we mentioned earlier, the necessary condition is the rank of Kraus-operator M_s be 1:

$$M_s = \sqrt{a^{(s)}} |l^{(s)}\rangle\langle r^{(s)}| \ . \tag{22}$$

From eq.(1) it follows that the post-measurement state is always $|l^{(s)}\rangle$ independently of the otherwise unknown pre-measurement state. If we apply our general rule (13) to this trivial case we find that, indeed, the best estimate is the true one: $|\chi^{(s)}_{\rm post}\rangle = |l^{(s)}\rangle$. The rule (19) yields $|\chi^{(s)}_{\rm pre}\rangle = |r^{(s)}\rangle$ for the best estimate of the pre-measurement state. Hence the ultimate form of the rank-one Kraus-operators is this:

$$M_s = \sqrt{a^{(s)}} |\chi^{(s)}_{\rm post}\rangle\langle \chi^{(s)}_{\rm pre}| \ . \tag{23}$$

The corresponding effects E_s are then given by

$$E_s = a^{(s)} |\chi^{(s)}_{\rm pre}\rangle\langle \chi^{(s)}_{\rm pre}| \ . \tag{24}$$

The completeness relation $\sum E_s = \mathbb{1}$ constrains the pre-measurement state estimates to form an overcomplet basis in general. The set of post-measurement states is not constrained at all. Note that the multiplicity of different measurement results s may exceed the number d of levels in our system. Since neither $|\chi^{(s)}_{\rm pre}\rangle$ nor $|\chi^{(s)}_{\rm post}\rangle$ have to form orthogonal systems they are in general not the eigenstates of any Hermitian observable. So we are still having a generalized measurement and not a projective measurement. The post-measurement state is nevertheless exactly known ($G_{\rm post}=1$) and the optimal estimate of the pre-measurement state $|\chi^{(s)}_{\rm pre}\rangle$ is of maximal estimation fidelity $G_{\rm pre}=2/(d+1)$.

We turn to a further aspect of information gain and state disturbance. In eq.(5) we have uniquely decomposed the measurement operation M_s which corresponds to the measurement result s, into the positive operator $\sqrt{E_s}$ and a unitary operator U_s. The unitary part does not change the von Neumann entropy. By virtue of eq.(2), all information, which is contained in a measurement result, goes back to $\sqrt{E_s}$. In particular, the estimation fidelities $G_{\rm post}$ and $G_{\rm pre}$ (14,18) depend only on the eigenvalues of E_s. The part $\sqrt{E_s}$ of M_s represents, at a given information gain, the unavoidable minimal disturbance of the state vector. We call $\sqrt{E_s}$ the *pure measurement part* of a generalized measurement and a measurement with $U_s=\mathbb{1}$ a *pure measurement.* The operation fidelity F depends on the unitary parts U_s, too. The inequality (21) shows that the maximal operation fidelity F is limited by $G_{\rm post}$ and therefore by the pure measurement part.

Having connected the three mean fidelities $F, G_{\rm pre}$ and $G_{\rm post}$, we connect now the best guesses $|\chi^{(s)}_{\rm pre}\rangle$ and $|\chi^{(s)}_{\rm post}\rangle$ for the pre- and post-measurement states, respectively. The two best guesses are the distinguished pair of l.h.s. and r.h.s. eigenvectors to the same eigenvalue, cf. eqs.(13) and (19). Invoking the expansions (8) and (9), this leads directly to the results

$$U_s|\chi^{(s)}_{\rm pre}\rangle=|\chi^{(s)}_{\rm post}\rangle \tag{25}$$

and

$$\frac{M_s|\chi^{(s)}_{\rm pre}\rangle}{\sqrt{a^{(s)}_{\max}}}=|\chi^{(s)}_{\rm post}\rangle\ . \tag{26}$$

Equation (25) shows that the best estimate for the post-measurement state can be obtained from the best estimate of the pre-measurement state by applying merely the unitary part U_s of the measurement operator. This has the surprising consequence that for all pure measurements the best estimations for the pre- and post-measurement state always agree if the in-going state $|\psi\rangle$ is completely unknown. This is the case regardless of the values of the operation fidelity F and the estimation fidelities $G_{\rm pre}$ and $G_{\rm post}$.

Finally we give a physical interpretation of relation (26). As a matter of fact, both the pre- and post-measurement states become only partially revealed by the estimation procedure. Nonetheless, even non-optimal estimates $|\chi^{(s)}_{\rm pre}\rangle$, $|\chi^{(s)}_{\rm post}\rangle$ must obey the constraint (26) expressing the certain fact that the post-measurement state results from the generalized measurement (1) of the pre-measurement one. Recall that we estimated the optimum pre- and post-measurement states by maximizing independently the pre- and post-measurement fidelities. We did not guarantee explicitly that the two optimum states $|\chi^{(s)}_{\rm pre}\rangle$, $|\chi^{(s)}_{\rm post}\rangle$ satisfy the exact constraint. The derived result (26) proves that they do.

In conclusion, we have studied generalized measurement $\{M_s\}$ on a single d-level quantum system. For the case when the initial state is pure and otherwise completely unknown, we pointed out that the best estimates of the pre- and post-measurement states for a given measurement readout s are the respective right and left eigenvectors of M_s, belonging to the (common) largest eigenvalue. The mean post-measurement estimation fidelity of the measurement device is also calculated and shown to satisfy a simple relationship with the mean pre-measurement estimation fidelity. A constraint between the post-measurement estimation fidelity and the operation fidelity of the measurement illustrates how state disturbance and information gain about the post-measurement state are competing with each other. We have shown that for pure generalized measurements the independent best estimates of the pre- and post-measurement states agree. We have proved that, in general, they are related via the corresponding measurement operator as we expect of them.

This work was supported by the Optik Zentrum Konstanz. L.D. acknowledges the support from the Hungarian Science Research Fund under Grant 32640.

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